1. The log-likelihood function is $l(\theta) = -\sum_{1\le j\le n} \log\{1+(X_j-\theta)^2\} - n\log\pi$. Let $\dot{l}(\theta)=0$, leading to the equation that the MLE must satisfy:

$$s(\hat\theta) = \sum_{1\le j\le n} \frac{2(X_j-\hat\theta)}{1+(X_j-\hat\theta)^2} = 0. \tag{1}$$

Note

$$\dot s(\theta) \equiv \frac{\partial}{\partial\theta} s(\theta) = 2\sum_{j=1}^n \frac{1-(X_j-\theta)^2}{\{1+(X_j-\hat\theta)^2\}^2}.$$

Hence the Newton-Raphson iteration is defined as

$$\hat\theta_{k+1} = \hat\theta_k - s(\hat\theta_k)/\dot s(\hat\theta_k), \qquad k=1,2,\cdots.$$

Note that the score function $s(\theta)$ is not monotone in θ. Hence (1) may have more than one solutions. We should start the above Newton-Raphson iteration with a good initial value. Since the density $f(\cdot,\theta)$ is symmetric around θ, it makes sense to consider either the sample mean or sample median as an initial estimate. However $E(X_1)$ is not well-defined, so the sample mean may not be a good estimator θ. Thus we may use the sample median as the initial value for our algorithm.

2. Note that $P(u_j \le X_j \le v_j) = e^{-\lambda u_j} - e^{-\lambda v_j}$, and $f_{X_j}(x|u_j\le X_j\le v_j) = \lambda e^{-\lambda x}/\{e^{-\lambda u_j} - e^{-\lambda u_j}\}$. Hence,

$$\tilde X_j(\lambda) \equiv E_\lambda(X_j|X_j\in[u_j,v_j]) = \frac{1}{\lambda} + \frac{u_j\exp(-\lambda u_j) - v_j\exp(-\lambda v_j)}{\exp(-\lambda u_j)-\exp(-\lambda v_j)}.$$

The log-likelihood function based on the full sample is

$$l(\theta) \equiv l(\theta; X_1,\cdots,X_n) = n\log\lambda - \lambda\sum_{j=1}^n X_j,$$

which yields the MLE based on full sample $\hat\theta(X_1,\cdots,X_n) = n/\sum_{1\le j\le n} X_j$.

Now the E-step is

$$Q(\lambda) = E_{\lambda_0}\{l(\theta)\,|\,Y_j\in[u_j,v_j] \text{ for } m<j\le n\} = n\log\lambda - \lambda\sum_{i=1}^m X_i - \lambda\sum_{j=m+1}^n \tilde X_j(\lambda_0),$$

and the M-step is simply

$$\lambda_1 = n/\{\sum_{i=1}^m X_i + \sum_{j=m+1}^n \tilde X_j(\lambda_0)\}.$$

The EM-algorithm iterates E-step and M-step with, for example, initial value $\lambda_0 = m/(X_1+\cdots+X_m)$.

3. (a) Note $l(p) = X\log p + (n-X)\log(1-p)$, $s(p) = X/p - (n-X)/(1-p)$, and $\dot s(p) = -X/p^2 - (n-X)/(1-p)^2$. Hence the Fisher information is

$$\mathcal{I}(p) = -E_p\{\dot s(p)\} = n/p + n/(1-p) = n/\{p(1-p)\}.$$

The C-R lower bound for the variance of unbiased estimator of $\theta(=p^2)$ is $(\frac{d\theta}{dp})^2/\mathcal{I}(p) = 4p^3(1-p)/n$.

(b) Note $L(p) = \prod_{j=1}^n p^{X_j}(1-p)^{1-X_j}$. This yields $\hat p = X/n$, where $X = \sum_{j=1}^n X_j$. Hence $\hat\theta = (\hat p)^2 = X^2/n^2$.

(c) Note

$$E_p(X^2) = \sum_{i=1}^{n} E_p(X_i^2) + \sum_{1 \le i \ne j \le n} E_p(X_i X_j) = nE_p(X_1^2) + (n^2 - n)E_p(X_1 X_2) = np + (n^2 - n)p^2.$$

Hence $E_p(\hat{\theta}) = p^2 + p(1-p)/n \ne p^2$, i.e $\hat{\theta}$ is a biased estimator for θ with bias $p(1-p)/n$.

(d) We draw bootstrap sample $X_1^*, \cdots, X_n^*$ from Bernoulli distribution with probability $\hat{p}$. Define the bootstrap estimator $\hat{\theta}^* = (X_1^* + \cdots + X_n^*)^2/n^2$. The bootstrap estimator for the bias of $\hat{\theta}$ is $\text{Bias}^* \equiv E_{\hat{p}}(\hat{\theta}^*) - \hat{\theta}$. In practice, $E_{\hat{p}}(\hat{\theta}^*)$ may be estimated via repeated bootstrap samplings.

Note. For this simple example, the bias estimator admits a simple analytic formula $\text{Bias}^* = \hat{p}(1-\hat{p})/n$, which is the simple plug-in estimator.

4. Since both Null and Alternative Hypotheses are completely specified, by the Neyman-Pearson Lemma, with a sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the most powerful test will reject H_0 for large values of

$$LR = \frac{f_{\mathbf{X}}(\mathbf{x}; \theta_2)}{f_{\mathbf{X}}(\mathbf{x}; \theta_1)}.$$

Now,

$$LR = \frac{\frac{1}{(\sqrt{2\pi\theta_2})^n} \exp\left[-\frac{1}{2}\sum_{i=1}^n x_i^2/\theta_2\right]}{\frac{1}{(\sqrt{2\pi\theta_1})^n} \exp\left[-\frac{1}{2}\sum_{i=1}^n x_i^2/\theta_1\right]}$$

which reduces to

$$LR = \left(\sqrt{\theta_1/\theta_2}\right)^n \exp\left[-\frac{1}{2}\sum_{i=1}^n x_i^2/(1/\theta_2 - 1/\theta_1)\right].$$

Obviously, this rejects H_0 for small values of

$$\frac{1}{2}\sum_{i=1}^n x_i^2/(1/\theta_2 - 1/\theta_1).$$

So, if $\theta_1 > \theta_2$, the most powerful test rejects H_0 for small values of $\sum x_i^2$. If, on the other hand, $\theta_1 < \theta_2$, the most powerful test rejects H_0 for large values of $\sum x_i^2$.

The critical value depends on the distribution under H_0. For $n = 10$, $T = \sum X_i^2 \sim \chi^2_{10}$ under $H_0 : \theta = \theta_1 = 1$. To test against $H_1 : \theta = \theta_2 = 2$, we reject H_0 iff

$$T > \chi^2_{10,\alpha} = 18.30 \quad \text{for } \alpha = 0.05,$$

where $\chi^2_{n,\alpha}$ is the upper $100\alpha\%$ point of the χ^2 distribution with n degrees-of-freedom. The power is

$$P\{T > 18.30|\theta = 2\} = P\{T/2 > 9.15\} = P\{\chi^2_{10} > 9.15\} = 0.518.$$

5. Let $\sigma_1^2 > \sigma_0^2$. For simple hypotheses $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$, the MPT is defined in terms of the likelihood ratio statistic

$$\text{LR} = \frac{\sigma_1^{-n}\exp\{-\frac{1}{2\sigma_1^2}\sum_i X_i^2\}}{\sigma_0^{-n}\exp\{-\frac{1}{2\sigma_0^2}\sum_i X_i^2\}} = (\sigma_0/\sigma_1)^n \exp\left\{\frac{1}{2}(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2})\sum_i X_i^2\right\}.$$

Therefore LR$> K$ is equivalent to $\sum_i X_i^2 > K_1$. Let

$$P_{\sigma_0^2}\left\{\sum_{i=1}^n X_i^2 > K_1\right\} = P_{\sigma_0^2}\left\{\frac{1}{\sigma_0^2}\sum_{i=1}^n X_i^2 > K_1/\sigma_0^2\right\} \equiv \alpha,$$

Hence $K_1 = \sigma_0^2 \chi^2_{n,\alpha}$, where $\chi^2_{n,\alpha}$ is the upper α-point of χ^2-distribution with n degrees of freedom. Note this test does not depend on σ_1^2, and is therefore the MPT for any $\sigma_1^2 > \sigma_0^2$. On the other hand, for any $\sigma^2 < \sigma_0^2$,

$$P_{\sigma^2}\left\{\sum_{i=1}^n X_i^2 > \sigma_0^2 \chi^2_{n,\alpha}\right\} = P_{\sigma^2}\left\{\frac{1}{\sigma^2}\sum_{i=1}^n X_i^2 > \frac{\sigma_0^2}{\sigma^2}\chi^2_{n,\alpha}\right\} \le P_{\sigma^2}\left\{\frac{1}{\sigma^2}\sum_{i=1}^n X_i^2 > \chi^2_{n,\alpha}\right\} = \alpha.$$

Hence it is the desired UMPT.